Finite Math - Spring 2017
Lecture Notes - 4/12/2017

Homework

- Section 4.7 - 21, 22, 23, 25, 26, 35, 36, 37, 38, 39, 40

Section 4.7 - Leontief Input-Output Analysis

Two-Industry Model. To simplify the ideas, we will assume we are in an economy with only two industries: coal and steel. In this economy, to produce \$1 worth of coal requires an input of \$0.10 from the coal sector and \$0.20 from the steel sector; and to produce \$1 worth of steel requires an input of \$0.20 from the coal sector and \$0.40 from the steel sector. The final demand (the demand from all other users of coal and steel) is \$20 billion for coal and \$10 billion for steel. What we would like to know is how much total coal and steel needs to be produced to meet this final demand.

If the coal and steel sector produces just the final demand of coal and steel, it would require:

<u>Coal</u>

$$0.1(20) + 0.2(10) = \$4 \text{ billion of coal}$$

<u>Steel</u>

$$0.2(20) + 0.4(10) = \$8 \text{ billion of steel}$$

This only leaves \$16 billion of coal and \$2 billion of steel left over to meet that final demand, well below the required amounts.

So, we need to not only meet the final demand, but also the internal demand. To figure out how to do this, we need two variables

$$x = \text{total output from coal industry}$$

$$y = \text{total output from steel industry.}$$

Then the internal demands (amount of coal and steel required to produce x amount of coal and y amount of steel) are as follows:

<u>Coal</u>

$$0.1x + 0.2y \text{ internal demand for coal}$$

<u>Steel</u>

$$0.2x + 0.4y \text{ internal demand for steel}$$

So, we can create equations for the total amount of coal and steel required by adding the internal and final demands to get the system of equations:

$$\begin{array}{rcccl} \text{Total} & & \text{Internal} & & \text{Final} \\ \text{output} & & \text{demand} & & \text{demand} \\ x & = & 0.1x + 0.2y & + & 20 \\ y & = & 0.2x + 0.4y & + & 10 \end{array}$$

which we rewrite in matrix form as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 20 \\ 10 \end{bmatrix}$$

or using letters for the matrices

$$X = MX + D.$$

D is called the *final demand matrix*, X is called the *output matrix*, and M is called the *technology matrix*. The technology matrix should be read as the inputs entering from the left and outputs leaving from above. That is,

$$\begin{array}{ccc} & & \text{Output} \\ & & \begin{array}{cc} C & S \\ \uparrow & \uparrow \end{array} \\ \text{Input} & \begin{array}{c} C \rightarrow \\ \\ S \rightarrow \end{array} & \begin{bmatrix} \begin{pmatrix} \text{input from } C \\ \text{to produce \$1} \\ \text{of coal} \end{pmatrix} & \begin{pmatrix} \text{input from } C \\ \text{to produce \$1} \\ \text{of steel} \end{pmatrix} \\ \begin{pmatrix} \text{input from } S \\ \text{to produce \$1} \\ \text{of coal} \end{pmatrix} & \begin{pmatrix} \text{input from } S \\ \text{to produce \$1} \\ \text{of steel} \end{pmatrix} \end{bmatrix} = M \end{array}$$

(C stands for the coal industry and S for the steel industry).

Now that the individual pieces are understood, let's finish solving the problem. Let's begin by solving the matrix equation first:

$$\begin{array}{rcl} X & = & MX + D \\ X - MX & = & D \\ (I - M)X & = & D \\ X & = & (I - M)^{-1}D \end{array}$$

(Note that this solution requires $I - M$ to have an inverse!)

Now we actually work this out with the numbers from this problem

Solution.

$$
\begin{aligned}
I - M &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.9 & -0.2 \\ -0.2 & 0.6 \end{bmatrix} \\
(I-M)^{-1} &= \frac{1}{(0.9)(0.6) - (-0.2)(-0.2)} \begin{bmatrix} 0.6 & 0.2 \\ 0.2 & 0.9 \end{bmatrix} = \begin{bmatrix} 1.2 & 0.4 \\ 0.4 & 1.8 \end{bmatrix} \\
X = (I-M)^{-1}D &= \begin{bmatrix} 1.2 & 0.4 \\ 0.4 & 1.8 \end{bmatrix} \begin{bmatrix} 20 \\ 10 \end{bmatrix} = \begin{bmatrix} 28 \\ 26 \end{bmatrix}
\end{aligned}
$$

So to meet the internal and final demands, $28 *billion of coal and* $26 *billion of steel must be produced.*

To summarize, these are the steps to solving an input-output analysis problem:

(1) Find the technology matrix M and the final demand matrix D.

(2) Find $I - M$.

(3) Find $(I - M)^{-1}$.

(4) Find $X = (I - M)^{-1}D$.

(5) Interpret the answer in words.

Example 1. *The economy of a small island nation is based on two sectors, agriculture and tourism. Production of a dollar's worth of agriculture requires an input of* $0.20 *from agriculture and* $0.15 *from tourism. Production of a dollar's worth of tourism requires an input of* $0.40 *from agriculture and* $0.30 *from tourism. Find the output from each sector that is needed to satisfy a final demand of* $60 *million for agriculture and* $80 *million for tourism.*

Solution. $148 *million from agriculture and* $146 *million from tourism.*

Three-Industry Model. To solve one of these problems for any number of industries is done in the exact same way as the two-industry model; all we need to know is how each industry depends on the others.

Example 2. *An economy is based on three sectors: coal, oil, and transportation. Production of a dollar's worth of coal requires an input of* $0.20 *from the coal sector and* $0.40 *from the transportation sector. Production of a dollar's worth of oil requires an input of* $0.10 *from the oil sector and* $0.20 *from the transportation sector. Production of a dollar's worth of transportation requires an input of* $0.40 *from the coal sector,* $0.20 *from the oil sector, and* $0.20 *from the transportation sector. Find the output from each sector that is needed to satisfy a final demand of* $30 *billion for coal,* $10 *billion for oil, and* $20 *billion for transportation.*

Solution. *First identify M and D:*

$$M = \begin{bmatrix} 0.2 & 0 & 0.4 \\ 0 & 0.1 & 0.2 \\ 0.4 & 0.2 & 0.4 \end{bmatrix} \quad D = \begin{bmatrix} 30 \\ 10 \\ 20 \end{bmatrix}.$$

Then

$$I - M = \begin{bmatrix} 0.8 & 0 & -0.4 \\ 0 & 0.9 & -0.2 \\ -0.4 & -0.2 & 0.6 \end{bmatrix}$$

and

$$(I - M)^{-1} = \begin{bmatrix} 1.7 & 0.2 & 0.9 \\ 0.2 & 1.2 & 0.4 \\ 0.9 & 0.4 & 1.8 \end{bmatrix}.$$

So, the required output is

$$X = (I - M)^{-1}D = \begin{bmatrix} 1.7 & 0.2 & 0.9 \\ 0.2 & 1.2 & 0.4 \\ 0.9 & 0.4 & 1.8 \end{bmatrix} \begin{bmatrix} 30 \\ 10 \\ 20 \end{bmatrix} = \begin{bmatrix} 71 \\ 26 \\ 67 \end{bmatrix}$$

To meet the internal and final demands, the economy must produce \$71 *billion of coal,* \$26 *billion of oil, and* \$67 *billion for transportation.*

Example 3. *An economy is based on three sectors: agriculture, manufacturing, and energy. Production of a dollar's worth of agriculture requires an input of* \$0.20 *from the agriculture sector,* \$0.20 *from the manufacturing sector, and* \$0.20 *from the energy sector. Production of a dollar's worth of manufacturing requires an input of* \$0.40 *from the agriculture sector,* \$0.10 *from the manufacturing sector, and* \$0.10 *from the energy sector. Production of a dollar's worth of energy requires an input of* \$0.30 *from the agriculture sector,* \$0.10 *from the manufacturing sector, and* \$0.10 *from the energy sector. Find the output from each sector that is needed to satisfy a final demand of* \$10 *billion for agriculture,* \$15 *billion for manufacturing, and* \$20 *billion for energy.*

Solution. \$40.1 *billion from agriculture,* \$29.4 *billion from manufacturing, and* \$34.4 *billion from energy.*